Portfolio Optimization
Under a Stressed-Beta Model

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**Stressed-Beta Model (Under Physical Measure \( \mathcal{P} \))**

- \( M_t \) value of market at time \( t \)
- \( S_t \) value of asset at time \( t \)

\[
\frac{dM_t}{M_t} = \mu dt + \Sigma dW_t \\
\frac{dS_t}{S_t} = rdt + \beta(M_t) \left( \frac{dM_t}{M_t} - rdt \right) + \sigma dZ_t
\]

\( \beta(M_t) = \beta + \delta \mathbb{I}_{\{M_t < c\}} \)

Brownian motions \( W_t, Z_t \) independent
Why This Model?

It extends the most popular model in financial economics, CAPM.

CAPM criticism #1: nonlinear relationship between asset, market returns.

CAPM criticism #2: Beta is backward-looking (estimated using OLS regression).

Solution: Developed calibration technique to yield forward-looking parameters.

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The Optimization Problem

Agent observes level of stock market, invests in stock or riskless bond

Finite horizon $T$

$CRRA$ utility function $U(y) = \frac{y^\lambda}{\lambda}$

Objective: max expected terminal utility

Constraints: budget, self-financing trading strategies

Set of admissible strategies $\mathcal{A}$

Let $\pi_t$ and $1 - \pi_t$ denote fraction of wealth in stock, bond
Merton Problem

If $\beta(M_t) = \beta^H$, a constant, we have the Merton problem

Optimal Strategy:

$$\pi^* = \frac{(\mu - r)\beta^H}{(1 - \lambda)((\beta^H)^2 \Sigma^2 + \sigma^2)}.$$ 

This is a constant depending on model parameters, level of risk aversion.

Equation will appear again in similar form when we study Stressed-Beta optimization.
Wealth Dynamics

Consider wealth process $Y$. Initial endowment $y$

$$dY_t = Y_t \left[ (r + (\mu - r)\beta(M_t)\pi_t)\,dt + \beta(M_t)\Sigma\pi_t\,dW_t + \sigma\pi_t\,dZ_t \right]$$

Notice...no dependency on $S$, but dependency on $M$ through slope-switching mechanism.

Thus, we are dealing with a two-dimensional Markov process $(Y, M)$. 
The Value Function

Define expected terminal utility given starting conditions:

\[ J(t, y, m, \pi) = \mathbb{E} \left\{ \frac{Y_T^\lambda}{\lambda} \left| Y_t = y, M_t = m \right. \right\} \]

Define value function

\[ V(t, y, m) = \sup_{\pi \in \mathcal{A}} J(t, y, m, \pi) \]
The Value Function

Define expected terminal utility given starting conditions:

\[ J(t, y, m, \pi) = \mathbb{E} \{ Y_T^\lambda / \lambda \mid Y_t = y, M_t = m \} \]

Define value function

\[ V(t, y, m) = \sup_{\pi \in A} J(t, y, m, \pi) \]

\( V(t, y, m) \) satisfies stochastic Hamilton-Jacobi-Bellman (HJB) partial differential equation

\[
V_t + \sup_{\pi \in A} \left\{ y \left[ r + (\mu - r) \beta(m) \pi \right] V_y + m \mu V_m + \frac{1}{2} y^2 \pi^2 (\beta^2(m) \Sigma^2 + \sigma^2) V_{yy} \right. \\
+ \left. \frac{1}{2} m^2 \Sigma^2 V_{mm} + my \beta(m) \Sigma^2 \pi V_{my} \right\} = 0,
\]

terminal condition \( V(T, y, m) = y^\lambda / \lambda \)
Working with the PDE

Write the value function in separable form

\[ V(t, y, m) = \frac{y^\lambda}{\lambda} u(t, m) \]

Substituting into the HJB equation gives

\[
\begin{align*}
  u_t + \sup_{\pi \in \mathcal{A}} \left\{ \left[ r + (\mu - r)\beta(m)\pi \right] \lambda u + m \mu u_m \right. \\
  + \left. \frac{1}{2} \pi^2 (\beta^2(m)\Sigma^2 + \sigma^2) \lambda (\lambda - 1) u \\
  + \frac{1}{2} m^2 \Sigma^2 u_{mm} + m \beta(m)\Sigma^2 \pi \lambda u_m \right\} &= 0
\end{align*}
\]

with terminal condition \( u(T, m) = 1 \).
Optimal Fraction of Wealth in Stock

Quadratic form in $\pi$ attains a maximum at

$$\pi^*(m) = \frac{\beta(m)\Sigma^2}{(1 - \lambda)(\beta^2(m)\Sigma^2 + \sigma^2)} \frac{mu_m}{u} + \frac{(\mu - r)\beta(m)}{(1 - \lambda)(\beta^2(m)\Sigma^2 + \sigma^2)}$$

Second term in RHS has form of Merton

First term is “correction” to Merton. Since $u_m, u$ appear, need to keep working with PDE
Substituting $\pi^*$, 

$$
\begin{align*}
  u_t & \quad + \quad \frac{1}{2} \Sigma^2 m^2 u_{mm} + \left( m\mu + \frac{m\lambda(\mu - r)}{(1 - \lambda) \left( 1 + \frac{\sigma^2}{\beta^2(m)\Sigma^2} \right)} \right) u_m \\
  & \quad + \quad \left( r\lambda + \frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^2 \left( 1 + \frac{\sigma^2}{\beta^2(m)\Sigma^2} \right)} \right) u \\
  & \quad + \quad \frac{\lambda m^2 \Sigma^2}{2(1 - \lambda) \left( 1 + \frac{\sigma^2}{\beta^2(m)\Sigma^2} \right)} \frac{u_m^2}{u} = 0.
\end{align*}
$$
Working with the PDE

Form motivates approach taken in Zariphopoulou (2001) whereby soln posed in form of $u^\psi$, with $\psi$ referred to as distortion power.

In this case, approach doesn’t work, as value of $\psi$ needed to cause nonlinear term to vanish depends on market level.

Alternative approach: asymptotic expansion, expressing solution in terms of powers of $\sigma^2$ (stock noise)
Asymptotic Expansion

\[ u = u_0 + \sigma^2 u_1 + \mathcal{O}(\sigma^4), \]

Use first two terms \( u_0 + \sigma^2 u_1 \) to approximate solution

Substitute this expansion approx into the pde
PDE Given the Asymptotic Expansion

\[ \mathcal{L}_0 u_0 + \sigma^2 (\mathcal{L}_1 u_1 + s_0) = 0, \]

order-1 terms:

\[
\mathcal{L}_0 u_0 = u_{0t} + \frac{1}{2} \Sigma^2 m^2 u_{0mm} + \left( m \mu + \frac{m \lambda (\mu - r)}{1 - \lambda} \right) u_{0m} \\
+ \left( r \lambda + \frac{\lambda (\mu - r)^2}{2(1 - \lambda) \Sigma^2} \right) u_0 + \frac{\lambda m^2 \Sigma^2}{2(1 - \lambda)} \frac{u_{0m}^2}{u_0},
\]

order-\( \sigma^2 \) terms in \( u_1 \):

\[
\mathcal{L}_1 u_1 = u_{1t} + \frac{1}{2} \Sigma^2 m^2 u_{1mm} + \left( m \mu + \frac{m \lambda (\mu - r)}{1 - \lambda} \right) u_{1m} \\
+ \left( r \lambda + \frac{\lambda (\mu - r)^2}{2(1 - \lambda) \Sigma^2} \right) u_1,
\]
PDE Given the Asymptotic Expansion

Source term:

\[ s_0 = -\frac{m\lambda(\mu - r)}{(1 - \lambda)\beta^2(m)\Sigma^2}u_{0m} - \frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^4\beta^2(m)}u_0. \]
Solving the Order-1 PDE

Set $\mathcal{L}_0 u_0 = 0$

subject to terminal condition $u_0(T, m) = 1$

Since terminal condition is constant, and there are terms involving derivatives with respect to $m$, solution given by

$$u_0(t) = e^{R(T-t)}$$

where constant rate $R$ given by

$$R = r\lambda + \frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^2}$$

Note this PDE results if we assumed $\sigma = 0$ from the beginning
Solving the Order-$\sigma^2$ PDE

Set $\mathcal{L}_1 u_1 + s_0 = 0$

subject to terminal condition $u_1(T, m) = 0$

Then the PDE is

$$u_{1t} + \frac{1}{2} \sum m^2 u_{1mm} + \left( m\mu + \frac{m\lambda(\mu - r)}{1 - \lambda} \right) u_{1m}$$

$$+ Ru_1 - \frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^4 \beta^2(m)} u_0 = 0$$
Solving the Order-$\sigma^2$ PDE

Using the transformation $u_1(t, m) = w_1(t, m)e^{R(T-t)}$,

$$w_{1t} + \frac{1}{2}\Sigma^2 m^2 w_{1mm} + Bmw_1m + \frac{C}{\beta^2(m)} = 0$$

where

$$B = \mu + \frac{\lambda(\mu - r)}{1 - \lambda}$$

$$C = -\frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^4}$$

and the terminal condition is $w_1(T, m) = 0$
Solving the Order-$\sigma^2$ PDE

Use Feynman-Kac to write $w_1(t, m)$ as stochastic representation

$$w_1(t, m) = \mathbb{E} \left\{ \int_t^T \frac{C}{\beta^2(X_s)} ds \mid X_t = m \right\},$$

where $X_t$ is a stochastic process with dynamics:

$$dX_t = BX_t dt + \Sigma X_t dW_t.$$

Using slope-switching mechanism definition,

$$w_1(t, m) = \frac{C}{\beta^2} \left[ T - t - D \mathbb{E}_m \left\{ \int_t^T 1_{\{X_s < c\}} ds \right\} \right],$$

where $\mathbb{E}_m$ is shorthand for $\mathbb{E} \{ \cdot \mid X_t = m \}$,

$$D = \frac{\delta(2\beta + \delta)}{\beta^2 + \delta(2\beta + \delta)}$$
Solving the Order-$\sigma^2$ PDE

Term in expectation is occupation time of geometric Brownian motion

In the interest of time, we briefly outline the remaining steps:

• Consider log-process and apply Girsanov’s Theorem to remove drift

• Treat separately the cases $m = c$, $m < c$, and $m > c$
  (starting market level at, below, above boundary)
Solving the Order-\(\sigma^2\) PDE

Now that we have all of the pieces, we may compute \(u = u_0 + \sigma^2 u_1\)

Recall \(u_0(0) = e^{RT}\) (when we start at time \(t = 0\))

Correction term is \(u_1(0, m) = w_1(0, m)e^{RT}\)

where we have formulas for \(w_1(0, m)\) depending on whether \(m = c\), \(m < c\), or \(m > c\)
Optimal Fraction of Wealth in Stock

Returning to the optimal fraction of wealth $\pi^*(m)$ and substituting approximation $u = u_0 + \sigma^2 u_1$, we have

$$\pi^*(m) = \pi_0^*(m) + \sigma^2 \pi_1^*(m) + O(\sigma^4),$$

where

$$\begin{align*}
\pi_0^*(m) &= \frac{\mu - r}{(1 - \lambda)\beta(m)\Sigma^2}, \\
\pi_1^*(m) &= \frac{1}{\beta(m)(1 - \lambda)} \frac{mu_1 m}{u_0} - \frac{\mu - r}{(1 - \lambda)\beta^3(m)\Sigma^4}.
\end{align*}$$
Optimal Fraction of Wealth in Stock

\[ \pi^*(m) = \pi^*_0(m) + \sigma^2 \pi^*_1(m) + O(\sigma^4), \]

First-order term \( \pi^*_0(m) \) represents optimal fraction of wealth invested in stock for case \( \sigma = 0 \)

Term \( \pi^*_1(m) \) represents correction for case \( \sigma \) nonzero

Given expressions for \( u_0 \) and \( u_{1m} \) and ignoring higher-order terms, \( \pi^*(m) \) may be estimated as

\[ \pi^*(m) = \pi^*_0(m) + \sigma^2 \pi^*_1(m) \]
Numerical Illustration; Data

Consider major stress period: wake of Lehman Brothers failure

Portfolios containing Cisco systems, bond optimized according to:
1) Merton  2) Stressed-Beta framework

S&P 500 is used as proxy for market

1-mo LIBOR used for risk-free rate

Start Date: September 26, 2008   End Date: one month later

Start date marks beginning of major decline in CISCO’s stock price, and coincides with high trading volume of short-dated call options used for model calibration
Numerical Illustration; Parameters

Some params need to be estimated from historical data:

$$\mu \quad \Sigma \quad \beta^H \quad \sigma$$

Estimation Period: July 1, 2008 – September 26, 2008

$\beta^H$ estimated from OLS regression of excess stock returns on excess market returns

$\sigma$ estimated as root mean squared error from regression

Risk-aversion parameter $\lambda = 0.5$

Forward-looking parameters calibrated from option data: $c, \beta, \delta$
**Numerical Illustration; Parameter Estimates**

**Table 1: Parameter Estimates; 7/1/2008 – 9/26/2008**

<table>
<thead>
<tr>
<th>$\hat{\mu}$</th>
<th>$\hat{\Sigma}$</th>
<th>$\hat{\sigma}$</th>
<th>$\hat{\beta}^H$</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\delta}$</th>
<th>$\hat{c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-19.26%</td>
<td>29.79%</td>
<td>21.34%</td>
<td>1.11</td>
<td>0.40</td>
<td>0.90</td>
<td>1250</td>
</tr>
</tbody>
</table>
Figure 1: Portfolio Utility \((\lambda = 0.5)\). Additionally, terminal wealth of Stressed-Beta portfolio is 6% higher than terminal wealth of Merton portfolio.
THANK YOU!